Transposing, and substituting $\chi_1 I_1$ for $\int_0^P h_1\, dp_1$ etc. we obtain

$$\frac{\frac{1}{2}(A_{P,1}+A_{P,2}) - \pi r_1^2\left[1+\frac{P}{E}(3\sigma-1)-\frac{\delta r}{r}\right]}{\frac{1}{2}(A_{P,1}-A_{P,2}) - \pi r_1^2 \frac{\delta r}{r}} = -\frac{\left(\frac{\chi_2}{\chi_1}\frac{I_2}{I_1}+1\right)}{\left(\frac{\chi_2}{\chi_1}\frac{I_2}{I_1}-1\right)}. \qquad (5.5)$$

The right hand side of this equation is completely determined by χ_2/χ_1, the ratio of the cube roots of the measured rates of flow, and the ratio I_2/I_1. The quantity $(A_{P,1} - A_{P,2})$ occurring in the denominator of the left hand side may be determined by direct balancing of the two forms of the assembly against any third reference assembly, and the quantity $\delta r/r$ is established by diametral measurements on the two pistons. The small quantity $P\,(3\,\sigma - 1)/E$ is known with sufficient accuracy from the elastic constants of the material. Subject, therefore, to further examination of the factor I_2/I_1 equation (5.5) enables the quantity $(A_{P,1} + A_{P,2})/2$, i. e. the mean of the effective areas of the two forms of the assembly, to be determined, as a function of the applied pressure, from the experimental observations.

It is evident that the term $\pi r_1^2 \cdot \delta r/r$ occurring in the denominator of the left hand side should be identical with $(A_{0,1} - A_{0,2})$ and this may be checked directly from the experimental data. If, as may be the case, the difference $(A_{P,1} - A_{P,2})$ is independent of pressure, the denominator of the left hand side may be written more simply as $-(A_{0,1} - A_{0,2})/2$, but it cannot of course be assumed a priori that this condition will hold.

c) Treatment of the integral 'I'

In order to estimate the value of I some simplifying assumptions must be introduced if the theory is not to become unjustifiably complicated. From the second of equations (5.4) it is clear that we can calculate I if we can express h and η as functions only of p. As regards h, the justification for assuming that the part of $h(x)$ arising from distortion due to the pressure in the interspace between the piston and cylinder may be taken as proportional to the pressure $p(x)$ at the same position has already been discussed. Bearing in mind that we are not really interested in the absolute values of I_1 and I_2, but only in their ratio, this assumption is not likely to lead us far astray. As before, there is an additional component of h arising from the longitudinal thrust on the piston, which will be proportional to the total applied pressure, P. We therefore write

$$h = H + \nu P + \mu p \qquad (5.6)$$

where μ and ν are constants.

The coefficient of viscosity at constant temperature is certainly determined uniquely by the pressure and there is considerable evidence available from published measurements that the dependence may be represented reasonably closely by an exponential function, in other words that we may write

$$\eta = \eta_0\, e^{\alpha p} \qquad (5.7)$$

where α is a constant and η_0 is the value at zero (or atmospheric) pressure. This relation has been found to hold with fair accuracy for most oils of types likely to be used in conjunction with pressure balances, although it appears that there may be more pronounced departures in the case of some silicone fluids (BRIDGMAN 1952; Amer. Soc. Mech. Engrs. 1953; ZOLOTYKH 1960).

The evaluation of I in terms of the constants in equations (5.6) and (5.7) is now straightforward and, writing for brevity

$$c = (H + \nu P)\,\alpha/\mu\ ,$$

we obtain

$$I = (l\eta_0)^{\frac{1}{3}}\, P^{\frac{3}{2}}\, I'\ ,$$

where

$$I' = (\alpha P)^{\frac{1}{3}}\left(c + \frac{\alpha P}{2}\right) \left\{\begin{matrix} c^3 + 3c^2 + 6c + 6 \\ - e^{-\alpha P}[(c+\alpha P)^3 + 3(c+\alpha P)^2 + 6(c+\alpha P) + 6] \end{matrix}\right\}^{-\frac{1}{3}} \qquad (5.8)$$

This quantity may conveniently be represented as a family of graphs showing its dependence on $(H + \nu P)/\mu P$ for a suitable range of values of αP.

In order to apply equation (5.8) the values of $(H + \nu P)/\mu P$ and αP corresponding to the experimental points are required. Denoting by ϱ_P the ratio $\chi_2 I_2/\chi_1 I_1$ at a given applied pressure P, and by ϱ_0 the extrapolated value of ϱ_P corresponding to zero applied pressure, and using equation (5.6), we find that

$$\varrho_P = \frac{H_2 + \nu P + \mu P/2}{H_1 + \nu P + \mu P/2}\ ,$$

whence, after some reduction, we obtain the equations

$$\frac{H_1 + \nu P}{\mu P} = \frac{\varrho_0 - 1}{\varrho_0 - \varrho_P}\left(\frac{\nu}{\mu} + \frac{1}{2}\right) - \frac{1}{2} \qquad (5.9)$$

and

$$\frac{H_2 + \nu P}{\mu P} = \varrho_P\,\frac{\varrho_0 - 1}{\varrho_0 - \varrho_P}\left(\frac{\nu}{\mu} + \frac{1}{2}\right) - \frac{1}{2}\ . \qquad (5.10)$$

We do not need to know the values of μ and ν, but an approximate figure for ν/μ is required. From the elementary theory leading to equation (2.6) we easily find

$$\nu/\mu = -\,\sigma/2\ \text{(approx.)}$$

whence we obtain, with sufficient accuracy, $\nu/\mu = -0.15$. Initially, of course, we cannot actually use the ratio $\chi_2 I_2/\chi_1 I_1$ since I_2/I_1 has not yet been determined. In practice, therefore, we commence by using simply the experimental ratio χ_2/χ_1 to obtain a first approximation to the correction factor, and then if necessary proceed to a second approximation.

To derive the appropriate value of αP we again ignore initially the distinction between $I_1(P)$ and $I_2(P)$, and denoting the quantity $(\chi_2 - \chi_1)\,P^{-\frac{1}{3}}$ at the applied pressure P by $\Delta\chi(P)$, we obtain from equations (5.4) and (5.6)

$$\begin{aligned}\Delta\chi(P) &= P^{\frac{2}{3}}\,(H_2 - H_1)/I(P) \\ &= (H_2 - H_1)\,(l\eta_0)^{-\frac{1}{3}}/I'(P)\end{aligned}$$

whence, dividing the experimental value of $\Delta\chi(P)$ into the extrapolated value corresponding to zero applied pressure, we have

$$\Delta\chi(O)/\Delta\chi(P) = I'(P)/I'(O)\ .$$

Making use of the values of $(H + \nu P)/\mu P$ derived above we are now able to determine the value of αP which best fits the experimental data by simple inspection of graphs or tables of the function I'. In this case, again, a second approximation may be derived if necessary.

Having carried out the above procedures we are now in a position to determine the values of I_1 and I_2, and consequently the ratio I_2/I_1, corresponding to the actual experimental points, and then to calculate, using equation (5.5) of the previous section, the changes in the effective areas of the assemblies as a function of the applied pressure.

d) Experimental method

The experimental procedures used in that part of the flow method involving the direct comparison of effective areas by balancing need no further consideration as they are exactly the same as those previously described in section 3 f. In the measurement of the rate of flow of the pressure balance fluid use is made of a very simple device. With the balance operating on an otherwise leak-proof system the change in volume of the contained fluid due to flow through the interspace between the piston and cylinder is exactly compensated by the gradual descent of the piston, and the rate of fall of the latter is thus directly proportional to the rate of flow. All that is necessary therefore is to time the descent of the piston over a constant short distance, the measured time being inversely proportional to the rate of flow. In practice this was carried out by using an optical magnification system and measuring the time of descent over a distance of the order 1 mm by stopwatch, but if the method were to be used at all extensively a more sophisticated procedure could obviously be substituted, using, for example, a photoelectric recording arrangement.

The work has been carried out using two piston-cylinder assemblies, of nominal effective areas 0.05 and 0.02 in², covering respectively pressure ranges up to about 600 bars and 1500 bars. The transmitting fluid used was in each case a mixture of two mineral oils, known commercially as Diala and Talpa respectively.

Since the coefficient of viscosity is markedly dependent on temperature precautions had to be taken to ensure that the temperature of the piston-cylinder assembly remained as constant as possible during a series of measurements. As a fully temperature-controlled room was not available it was necessary to determine a temperature coefficient in order that each series of readings could be converted to a common temperature, which was taken to be 20 °C. In general the apparatus and air temperatures were held to within a few tenths of a degree during any one series. In order to avoid extraneous friction, all the measurements were made with the piston and associated load in free rotation, the speed chosen being in the range 30 to 40 rev/min. In any group of measurements at a given pressure readings were taken alternately for the two directions of rotation, and the mean taken, to ensure that any possible effects due to small helical errors on the piston surface were eliminated.

The changes of effective areas with pressure were also measured, using the same pressure transmitting fluid in each case, by the similarity method. The results of the measurements, and the comparison of the two methods, are discussed in the next section.

6. Results of the Flow Method

a) Experimental parameters and correction terms

The various parameters and correction terms required in the derivation of the changes of effective area as a function of pressure are given in Tab. 3 for the two assemblies concerned, together with the distortion coefficients determined by the flow and similarity methods.

Table 3. *Parameters and correction terms*

Nominal area of assembly	Mean difference of piston diameters	$\frac{A_{0(1)} - A_{0(2)}}{A_0} = \frac{\delta r}{r}$ calculated from difference of piston diameters (parts in 10^5)	$\frac{A_{P(1)} - A_{P(2)}}{A_0}$ experimental value (parts in 10^5)	ϱ_0 (extrap)	Typical values of correction term I_2/I_1		Estimated value of α (bar^{-1})	Distortion coefficient (bar^{-1})	
					Pressure (bar)	I_2/I_1		Flow method	Similarity method
0.05 in²	5.45×10^{-5} in	21.6	21.4	1.48	0	1.000	3.2×10^{-3}	$4.2_5 \times 10^{-7}$	$4.3_8 \times 10^{-7}$
(0.322 cm²	($13.8_4 \times 10^{-5}$ cm)		(independent		140	1.002			
approx.)			of pressure)		280	1.005_5			
					560	1.004			
0.02 in²	2.75×10^{-5} in	17.5	17.5 (P = 0)	1.32	0	1.000	2.3×10^{-3}	$4.0_7 \times 10^{-7}$	$3.9_6 \times 10^{-7}$
(0.129 cm²	($6.9_8 \times 10^{-5}$ cm)		decreasing		250	1.001_5			
approx.)			smoothly to		500	1.003			
			15·0 (P=1500)		1000	1.002			
					1500	0.999			

A good check of the internal consistency of the different measurements is provided by a comparison of the figures in the third and fourth columns of the Table from which it will be seen that the changes of effective area at zero pressure, calculated from the measured piston diameters, are in very close agreement with the changes determined experimentally by direct balancing. The correction terms I_2/I_1, of which typical values are given, nowhere differ from unity by as much as 1% in the present range of experiments, but owing to the form of the right hand side of equation (5.5) they are just sufficiently significant to warrant taking them into account. If the flow method were to be extended to higher pressure ranges it is likely that larger corrections would be involved, and these might eventually limit the pressure range attainable with reliability.

It will be seen that somewhat different values of the coefficient α were found in the two cases. This